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THE EXTENDIBILITY OF DIOPHANTINE PAIRS WITH FIBONACCI NUMBERS AND SOME CONDITIONS

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Abstract. A set $\{a_1, a_2, \cdots, a_m\}$ of positive integers is called a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$. Let F_n be the nth Fibonacci number which is defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. In this paper, we find the extendibility of Diophantine pairs $\{F_{2k}, b\}$ with some conditions.

1. Introduction

A Diophantine m-tuple is a set which consists of m distinct positive integers satisfy the property that the product of any two of them is one less than a perfect square. If the set which consists of rational numbers satisfies the same property then we called rational Diophantine m-tuple. Diophantus found the first rational Diophantine quadruple $\{1/16, 33/16, 17/4, 105/16\}$. However, the first set of four positive integers with the property $\{1, 3, 8, 120\}$ was found by Fermat. Many famous mathematicians made lots of results related to the problems of Diophantine m-tuple, but still there are many open problems. Especially, the most famous problem is the extendibility of Diophantine m-tuple.

For any Diophantine triple $\{a, b, c\}$ with $a < b < c$, the set $\{a, b, c, d_{\pm}\}$ is a Diophantine quadruple, where

$$d_{\pm} = a + b + c + 2abc \pm 2rst$$

and r, s, t are the positive integers satisfying

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

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A folklore conjecture is that there does not exist a Diophantine quintuple. Recently, the conjecture has been proved by B. He, A. Togbé and V. Ziegler[12]. The stronger version of this conjecture states that if $\{a, b, c, d\}$ is a Diophantine quadruple and $d > \max\{a, b, c\}$ then $d = d_+$. These Diophantine quadruples are called regular.

We can find the importance of the extendiblity of Diophantine m-tuples in relation to the elliptic curves. We have to solve the equations

$$ax + 1 = \square,\ bx + 1 = \square,\ cx + 1 = \square$$

to extend the Diophantine triple $\{a, b, c\}$ to the Diophantine quadruple. Then we have the equation

$$E : y^2 = (ax + 1)(bx + 1)(cx + 1),$$

which is the elliptic curve by the product of three equations. We always have the integer points

$$(0, \pm 1),\ (d_+, \pm(at+rs)(bs+rt)(cr+st)),\ (d_-, \pm((at-rs)(bs-rt)(cr-st))),$$

and also $(-1, 0)$ if $1 \in \{a, b, c\}$ on E. For example, A. Dujella[3] proved that the elliptic curve

$$E_k : y^2 = ((k-1)x + 1)((k+1)x + 1)(4kx + 1)$$

has four integer points

$$(0, \pm 1),\ (16k^3 - 4k, \pm(128k^6 - 112k^4 - 20k^2 - 1))$$

under assumption that $\operatorname{rank}(E_k(\mathbb{Q})) = 1$. Similar results like [5] and [11] were proved for the equation

$$y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4} + 1)$$

and

$$y^2 = (F_{2k+1}x + 1)(F_{2k+3}x + 1)(F_{2k+5} + 1),$$

respectively, where F_n is the n-th Fibonacci number, which is defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

In 1977, Hoggatt and Bergum conjectured that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ is a Diophantine quadruple then d is a unique[13]. In 1999, A. Dujella proved this conjecture[2]. Furthermore, A. Filipin, Y. Fujita and A. Togbé proved that Diophantine pairs $\{F_{2k}, F_{2k+2}\}$ can be extended to Diophantine quintuples[9]. Recently, the extendibility of Diophantine pairs $\{F_{2k}, F_{2k+4}\}$ was proved by the author[14]. The Diophantine pairs $\{F_{2k}, F_{2k+4}\}$ has the ideal lower bound which is used in the Theorem of Baker and Wüstholz, since $F_{2k} + F_{2k+4} = 3F_{2k+2}$, that is $a + b = 3r$.

In this paper, we prove the extendibility of Diophantine pairs $\{F_{2k}, b\}$, where r is a divisor of $a+b$ and $b \leq 8a$. More precisely, the Diophantine triple $\{F_{2k}, b, c_1^+\}$ can be extended only to regular.

2. Preliminaries

2.1. The bounds of each elements of Diophantine triple

We can find the lower bounds of second element of the Diophantine triple $\{a, b, c\}$ with $a < b$ using the following lemma.

LEMMA 2.1. [9, Lemma 1.3] *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$.*

1. *If $b < 2a$, then $b > 2.1 \cdot 10^4$.*
2. *If $2a \leq b \leq 8a$, then $b > 1.3 \cdot 10^5$.*
3. *If $b > 8a$, then $b > 2 \cdot 10^3$.*

Let $\{a, b, c\}$ be a Diophantine triple, and r, s, t be the positive integers satisfying $ab + 1 = r^2,\ ac + 1 = s^2,\ bc + 1 = t^2$. Then we have

$$at^2 - bs^2 = a - b.$$

We easily find the form of solutions of the equation above is

$$(t\sqrt{a} + s\sqrt{b}) = (t_0\sqrt{a} + s_0\sqrt{b})(r + \sqrt{bc})^\nu.$$

If (t_0, s_0) belongs to the same class as either of the solutions $(\pm 1, 1)$ then s can be expressed as $s = s_\nu^\tau$, where $\tau \in \{\pm 1\}$ and

$$s_0 = s_0^\tau = 1,\ s_1^\tau = r + \tau a,\ s_{\nu+2}^\tau = 2rs_{\nu+1}^\tau - s_\nu^\tau.$$

Define $c_\nu^\tau = ((s_\nu^\tau)^2 - 1)/a$. Then, we obtain

$$c = c_\nu^\tau = \frac{1}{4ab}[(a + b \pm 2\sqrt{ab})(2ab + 1 + 2r\sqrt{ab})^\nu$$
$$+(a + b \mp 2\sqrt{ab})(2ab + 1 - 2r\sqrt{ab})^\nu - 2(a + b)].$$

First, we have the form of third element c in the Diophantine triple $\{a, b, c\}$ by the following theorem.

LEMMA 2.2. [8, Lemma 4.1] *Let $\{a, b, c\}$ be a Diophantine triple. Assume that $a < b \leq 8a$. Then $c = c_\nu^\tau$ for some ν and τ.*

Next, the following theorem gives us the bound of third element c in the Diophantine triple $\{a, b, c\}$.

Theorem 2.3. [8, Theorem 1.2] *Let* $\{a, b, c\}$ *be a Diophantine triple with* $a < b$. *Suppose that* $\{a, b, c, d\}$ *is a Diophantine quadruple with* $d > d_+$ *and that* $\{a, b, c', c\}$ *is not a Diophantine quadruple for any* c' *with* $0 < c' < d_-$, *where* d_+ *and* d_- *are defined by*

$$d_\pm = a + b + c + 2abc \pm 2rst,$$

respectively.

1. *If* $b < 2a$, *then* $c < b^6$.
2. *If* $2a \leq b \leq 8a$, *then* $c < 9.5b^4$.
3. *If* $b > 8a$, *then* $c < b^5$.

If $c = c_\nu^\tau$ then we can find the upper bound of c more specific by the following theorem.

Theorem 2.4. [9, Theorem 1.4] *Suppose that* $\{a, b, c_\nu^\tau, d\}$ *is a Diophantine quadruple with* $d > c_{\nu+1}^\tau$ *and that* $\{a, b, c', c_\nu^\tau\}$ *is not a Diophantine quadruple for any* c' *with* $0 < c' < c_{\nu-1}^\tau$.

1. *If* $b < 2a$, *then* $c \leq c_3^+$.
2. *If* $2a \leq b \leq 8a$, *then* $c \leq c_2^+$.

2.2. The Properties of solutions of Pell equation

We have to solve the system

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2$$

to extend the Diophantine triple $\{a, b, c\}$ to the Diophantine quadruple $\{a, b, c, d\}$. One can eliminate d to obtain the following system of Pell equations

$$ay^2 - bx^2 = a - b, \tag{2.1}$$

$$az^2 - cx^2 = a - c, \tag{2.2}$$

$$bz^2 - cy^2 = b - c. \tag{2.3}$$

Lemma 2.5. [4, Lemma 1] *There exist positive integers* i_0, j_0 *and integers* $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}, i = 1, \ldots, i_0, j = 1, \ldots, j_0$, *with the following properties:*

1. $(z_0^{(i)}, x_0^{(i)})$ *and* $(z_1^{(j)}, y_1^{(j)})$ *are solutions of (2.2) and (2.3), respectively.*
2. $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$ *satisfy the following inequalities*

$$0 < x_0^{(i)} \leq \sqrt{\frac{a(c-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}} < \sqrt[4]{ac},$$

$$0 \le |z_0^{(i)}| \le \sqrt{\frac{(s-1)(c-a)}{2a}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < \frac{c}{2},$$

$$0 < y_1^{(j)} \le \sqrt{\frac{b(c-b)}{2(t-1)}} < \sqrt{\frac{t+1}{2}} < \sqrt[4]{bc},$$

$$0 \le |z_1^{(j)}| \le \sqrt{\frac{(t-1)(c-b)}{2b}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < \frac{c}{3}.$$

3. *If (z,x) and (z,y) are positive integers of (2.2) and (2.3), respectively then there exist $i \in \{1,\dots,i_0\}$, $j \in \{1,\dots,j_0\}$ and integers $m,n \ge 0$ such that*

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})(s+\sqrt{ac})^m, \tag{2.4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t+\sqrt{bc})^n. \tag{2.5}$$

From now on, we omit the superscripts (i) and (j). By (2.4), we may write $z = v_m$, where

$$v_0 = z_0,\ v_1 = sz_0 + cx_0,\ v_{m+2} = 2sv_{m+1} - v_m, \tag{2.6}$$

and by (2.5), we may write $z = w_n$, where

$$w_0 = z_1,\ w_1 = tz_1 + cy_1,\ w_{n+2} = 2tw_{n+1} - w_n. \tag{2.7}$$

Lemma 2.6. [6, Lemma 3] *If $v_m = w_n$ then $n-1 \le m \le 2n+1$.*

2.3. Congruence relation between solutions of Pell equations

In this section, we give the congruence relations between v_m and w_n, and properties of initial terms of (2.6) and (2.7).

Lemma 2.7. [4, Lemma 4] *We have the following properties of v_m and w_n.*

$$v_{2m} \equiv z_0 + 2c[az_0m^2 + sx_0m] \pmod{8c^2},$$
$$v_{2m+1} \equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] \pmod{4c^2},$$
$$w_{2n} \equiv z_1 + 2c[bz_1n^2 + ty_1n] \pmod{8c^2},$$
$$w_{2n+1} \equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] \pmod{4c^2}.$$

We have a question such that when does $v_m = w_n$ have a solution and if there exists a solution of $v_m = w_n$ then which values are possible for the solution. The following lemma gives us the answer.

Lemma 2.8. [6, Lemma 8] *We have the following results.*

1. *If the equation $v_{2m} = w_{2n}$ has a solution then $z_0 = z_1$. Furthermore, $|z_0| = 1$ or $|z_0| = cr - st$ or $|z_0| < \min\{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\}$.*

2. *If the equation* $v_{2m+1} = w_{2n}$ *has a solution then* $|z_0| = t, |z_1| = cr - st$ *and* $z_0 z_1 < 0$.
3. *If the equation* $v_{2m} = w_{2n+1}$ *has a solution then* $|z_0| = cr - st, |z_1| = s$ *and* $z_0 z_1 < 0$.
4. *If the equation* $v_{2m+1} = w_{2n+1}$ *has a solution then* $|z_0| = t, |z_1| = s$ *and* $z_0 z_1 > 0$.

Furthermore, the solution of $v_m = w_n$ is more specific when $c = c_\nu^\tau \leq c_3^+$ by the following lemma.

LEMMA 2.9. [9, Lemma 3.1] *Assume that* $a < b \leq 8a$.

1. *Assume that* $b < 3a$. *In the case of* $c = c_1^-$, *we have* $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ *and* $v_{2m+1} \neq w_{2n+1}$. *Moreover, if* $v_{2m} = w_{2n}$ *then* $z_0 = z_1 = 1$.

2. *In the case of* $c = c_1^+$, *we have* $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ *and* $v_{2m+1} \neq w_{2n+1}$. *Moreover, if* $v_{2m} = w_{2n}$ *then* $z_0 = z_1$ *and* $|z_0| = 1$.

3. *In the case of* $c = c_2^-$, *we have* $v_{2m+1} \neq w_{2n}$ *and* $v_{2m+1} \neq w_{2n+1}$. *Moreover, we have the following:*
   (a) *If* $v_{2m} = w_{2n}$ *then* $z_0 = z_1$ *and* $|z_0| = 1$ *or* $cr - st$.
   (b) *If* $v_{2m} = w_{2n+1}$ *then* $|z_0| = cr - st$ *and* $|z_1| = s$ *with* $z_0 z_1 < 0$.
   Furthermore, (b) occurs if and only if (a) with $|z_0| = cr - st$ *occurs.*

4. *In the case of* $c \in \{c_2^+, c_3^-, c_3^+\}$, *we have* $v_{2m+1} \neq w_{2n}$ *and* $v_{2m} \neq w_{2n+1}$. *Moreover, we get the following:*
   (a) *If* $v_{2m} = w_{2n}$ *then* $z_0 = z_1$ *and* $|z_0| = 1$.
   (b) *If* $v_{2m+1} = w_{2n+1}$ *then* $|z_0| = t$ *and* $|z_1| = s$ *with* $z_0 z_1 > 0$.

2.4. Some theorems for applying the reduction method

From (2.4), (2.5) and sum of their conjugate, respectively, we get

$$v_m = \frac{1}{2\sqrt{a}}[(z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m + (z_0\sqrt{a} - x_0\sqrt{c})(s - \sqrt{ac})^m],$$
$$w_n = \frac{1}{2\sqrt{b}}[(z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n + (z_1\sqrt{b} - y_1\sqrt{c})(t - \sqrt{bc})^n].$$

Hence, we transform the equation $v_m = w_n$ into the following inequality.

LEMMA 2.10. [4, Lemma 5] *Assume that* $c > 4b$. *If* $v_m = w_n$ *and* $m, n \neq 0$ *then*

$$0 < m \log(s+\sqrt{ac}) - n \log(t+\sqrt{bc}) + \log \frac{\sqrt{b}(x_0\sqrt{c} + z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} + z_1\sqrt{b})} < \frac{8}{3}ac(s+\sqrt{ac})^{-2m}.$$

We use the following theorem and lemma to obtain the upper bound for m.

THEOREM 2.11. [1, p.20] *For a linear form*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_l \log \alpha_l \neq 0$$

in logarithms of l algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$ with rational coefficients $\beta_1, \beta_2, \ldots, \beta_l$, we have

$$\log |\Lambda| \geq -18(l+1)! l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log \beta,$$

where $\beta := \max\{|\beta_1|, \ldots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \cdots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

LEMMA 2.12. [7, Lemma 5] *Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

1. *If $\epsilon > 0$ then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AB^{-m} \tag{2.8}$$

   in integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

2. *Let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $p - q + r = 0$ then there is no solution of inequality (2.8) in integers m and n with*

$$\max\left\{\frac{\log(3Aq)}{\log B}, 1\right\} < m \leq M.$$

3. The extendibility of $\{F_{2k}, b\}$ with some conditions

Let a denote a F_{2k}, and we use this notation in the rest of the paper. In this section, we consider the extendibility of Diophantine triple $\{a, b, c_1^+\}$, where $b \leq 8a$ and r is a divisor of $a + b$. Let $a + b = \rho \cdot r$, where ρ is an integer. Then we get a bound of ρ such that $1 \leq \rho \leq 8$, since $a < r$ and $b < 8r$.

- If $\rho = 1$ then it is possible only for $a = 1$. However, it means that $b = 0$, which is a contradiction.

- If $\rho = 2$ then we get $b = a + 2$, and this case was proved by Fujita [10].
- If $\rho = 3$ then it is the case of $b = F_{2k+4}$ which was proved in [14].

Hence, we may assume that $\rho \geq 4$.

3.1. Bounds for m and k

LEMMA 3.1. *Suppose that $m, n \geq 2$. Then*

$$m \geq \frac{\sqrt{2a+1}-1}{2}.$$

Proof. For the case of c_1^+, we have

$$s_1^+ = a + r \equiv a \pmod{r} \quad \text{and} \quad t_1^+ = b + r \equiv b \pmod{r}.$$

Using the Lemma 2.7, we have

$$\pm am^2 + am \equiv \pm bn^2 + bn \pmod{r}.$$

Since r is a divisor of $a + b$ and $\gcd(a, r) = 1$, we have

$$m^2 + n^2 \pm m \pm n \equiv 0 \pmod{r}.$$

However, $2m^2 + 2m \geq m^2 + n^2 \pm m \pm n > 0$. Hence, we have

$$2(m^2 + m) \geq r > a.$$

□

We have the following inequality for linear form in logarithms.

LEMMA 3.2. *If $v_{2m} = w_{2n}$ with c_1^+ and $m, n \neq 0$ then*

$$\begin{aligned} 0 &< 2m \log(s + \sqrt{ac}) - 2n \log(t + \sqrt{bc}) + \log \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})} \\ &< 3.08(s + \sqrt{ac})^{-4m}. \end{aligned}$$

Proof. Put

$$P = \frac{1}{\sqrt{a}}(x_0\sqrt{c} + z_0\sqrt{a})(s + \sqrt{ac})^m, \; Q = \frac{1}{\sqrt{b}}(y_1\sqrt{c} + z_1\sqrt{b})(t + \sqrt{bc})^n.$$

Then

$$P^{-1} = \frac{\sqrt{a}(x_0\sqrt{c} - z_0\sqrt{a})}{c-a}(s - \sqrt{ac})^m, \; Q^{-1} = \frac{\sqrt{b}(y_1\sqrt{c} - z_1\sqrt{b})}{c-b}(t - \sqrt{bc})^n.$$

Therefore, the relation $v_m = w_n$ becomes

$$P - \frac{c-a}{a}P^{-1} = Q - \frac{c-b}{b}Q^{-1}.$$

Since $P > 0,\ Q > 0$ and

$$P - Q > \frac{c-a}{a}(Q-P)P^{-1}Q^{-1},$$

it follows that $P > Q$. Furthermore, we have

$$\frac{P-Q}{P} < \frac{c-a}{a}P^{-2} < \frac{1}{a(c-a)} \leq \frac{1}{39}.$$

Hence,

$$\begin{aligned} 0 < \log\frac{P}{Q} &= -\log(1-\frac{P-Q}{P}) < \frac{40}{39}(\frac{c-a}{a})P^{-2} \\ &< \frac{40}{39}\frac{c-a}{(\sqrt{c}-\sqrt{a})^2}(s+\sqrt{ac})^{-2m}. \end{aligned}$$

Since $c = c_1^+ = a + b + 2r > 4a$, we have $\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} < 3$. Hence, we obtain the result. □

According to Lemma 2.10, Lemma 3.2 and Theorem 2.11, we have $l = 3, d = 4, \beta = 2m$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_3 = \frac{(\sqrt{c} \pm \sqrt{a})\sqrt{b}}{(\sqrt{c} \pm \sqrt{b})\sqrt{a}}.$$

Let α_3' and α_3'' be the conjugates of α_3 whose absolute values are greater than one. Then

$$h'(\alpha_1) = \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t),$$

$$\begin{aligned} h'(\alpha_3) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_3\alpha_3'\alpha_3'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}(\sqrt{c}+\sqrt{a})(\sqrt{c}+\sqrt{b})(c-a))\} < \log(1.42c) \end{aligned}$$

and

$$\log|\Lambda| \geq -18\cdot 4!\ 3^4(32\cdot 4)^5\frac{1}{2}\log(2s)\frac{1}{2}\log(2t)\log(1.42c)\cdot\log(24)\cdot\log(2m).$$

Since

$$\log(\frac{8}{3}ac(s+\sqrt{ac})^{-4m}) < (-2m+1)\log(4ac)$$

and

$$\log(3.08(s+\sqrt{ac})^{-4m}) < (-2m+1)\log(4ac),$$

$c < 15a$ imply the following inequality

$$\frac{2m-1}{\log(2m)} < 9.556\cdot 10^{14}\log(30a)\log(21.3a). \tag{3.1}$$

By the lower bound of m and (3.1), we get $a < 9.35 \cdot 10^{40}$ and $c < 1.41 \cdot 10^{42}$. Since $(1.618)^{2k} < \alpha^{2k} < \overline{\alpha}^{2k} + \sqrt{5} \cdot (9.35 \cdot 10^{40})$, we get $k \leq 98$. Also, from (3.1) and the upper bound of a, we obtain $m < 2.17 \cdot 10^{20}$.

3.2. The reduction method

We can obtain an upper bound of m using the Lemma 2.12 with the inequalities

$$0 < m_1\kappa - n_1 + \mu_1 < A_3 B^{m_1},$$

where $m_1 := 2m,\ n_1 := 2n$ and

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2},\ \mu_1 = \frac{\log \alpha_3}{\log \alpha_2},\ A_3 = \frac{3.08}{\log \alpha_2},\ B = \alpha_1^2.$$

We apply the Lemma 2.12 to the logarithmic inequalities with $M := 2m \leq 4.34 \cdot 10^{20}$. We have to examine $10 \cdot 98 = 980$ cases. The program was developed in **PARI/GP** running with 70 digits. For the computations, if the first convergent such that $q > 6M_i$ with $i = 1, 2$ does not satisfy the condition $\epsilon > 0$ then we use the next convergent until we find the one that satisfies the conditions. Then we have the following Table 1 as results.

TABLE 1. Results from **PARI/GP** running

Case of ρ	Initial values	Use the next convergent
4	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	80 cases ($k = 19, \dots, 98$)
5	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	81 cases ($k = 18, \dots, 98$)
6	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	81 cases ($k = 18, \dots, 98$)
7	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	81 cases ($k = 18, \dots, 98$)
8	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	82 cases ($k = 17, \dots, 98$)

However, in all cases except the case of $\rho = 4$, we get $m \leq 6$. Hence, we take $M = 12$, and run the program again, then we obtain $m \leq 1$. When the case of $\rho = 4$, we get $m \leq 7$, so we take $M = 14$. Then also we obtain $m \leq 1$. Therefore, we have the following theorem.

THEOREM 3.3. *Let* $a = F_{2k}$, $a < b \leq 8a$ *and* $\{a, b, c_1^+, d\}$ *be a Diophantine quadruple with* $c_1^+ < d$. *If* r *is a divisor of* $a + b$ *then* $d = c_2^+$.

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